Reversible Jump MCMC

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Simple example

$$\theta \longrightarrow (\theta_1, \theta_2)$$

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Move has to be reversible:

$$u \sim Q(u)$$

$$(\theta, u) \longleftrightarrow (\theta_1, \theta_2)$$

Birth move

$$\theta_1 = \theta + u$$

$$\theta_2 = \theta - u$$

Birth move

$$\theta_1 = \theta + u$$

$$\theta_2 = \theta - u$$

Death move

$$\theta = \frac{\theta_1 + \theta_2}{2}$$

$$u = \frac{\theta_1 - \theta_2}{2}$$

Acceptance probability

likelihood ratio

$\times$

prior ratio

$\times$

inverse proposal probability ratio

likelihood ratio

$\times$

prior ratio

$\times$

inverse proposal probability ratio

$\times$

Jacobian

Acceptance probability

$$\frac{P(D|\theta_1,\theta_2)}{P(D|\theta)} \times \frac{P(\theta_1,\theta_2)}{P(\theta)} \times \frac{P(death)}{P(birth)Q(u)} \times \text{Jacobian}$$

Birth move

$$\theta_1 = \theta + u$$

$$\theta_2 = \theta - u$$

Birth move

$$\theta_1 = \theta + u$$

$$\theta_2 = \theta - u$$

Jacobian

$$\left| \begin{matrix} \frac{\partial \theta_1}{\partial \theta} & \frac{\partial \theta_1}{\partial u} \\ \frac{\partial \theta_2}{\partial \theta} & \frac{\partial \theta_2}{\partial u} \end{matrix} \right| = \left| \begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix} \right| = 2$$

Acceptance probability

$$\frac{P(D|\theta_1,\theta_2)}{P(D|\theta)} \times \frac{P(\theta_1,\theta_2)}{P(\theta)} \times \frac{P(death)}{P(birth)Q(u)} \times 2$$

Application to HMMs

Birth move

$$\mathbf{T} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

Birth move

$$\mathbf{T} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

$$\mathbf{T} \rightarrow \begin{pmatrix} \theta_{11} & \theta_{12} & \\ \theta_{21} & \theta_{22} & \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Birth move

$$\mathbf{T} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

$$\mathbf{T} \rightarrow \begin{pmatrix} \theta_{11} & \theta_{12} & w_1 \\ \theta_{21} & \theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Birth move

$$\mathbf{T} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

$$\mathbf{T} \rightarrow \begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Birth move

$$\mathbf{T} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$$

$$\tilde{\mathbf{T}} = \begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (1 - w_1)\theta_{11} & (1 - w_1)\theta_{12} & w_1 \\ (1 - w_2)\theta_{21} & (1 - w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

$$
\begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix} =
$$

$$
\begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}
$$

Jacobian

$$\begin{vmatrix}
\frac{\partial\tilde{\theta}_{11}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{11}}{\partial u1} & \frac{\partial\tilde{\theta}_{11}}{\partial u2} & \frac{\partial\tilde{\theta}_{11}}{\partial u3} & \frac{\partial\tilde{\theta}_{11}}{\partial w1} & \frac{\partial\tilde{\theta}_{11}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{12}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{12}}{\partial u1} & \frac{\partial\tilde{\theta}_{12}}{\partial u2} & \frac{\partial\tilde{\theta}_{12}}{\partial u3} & \frac{\partial\tilde{\theta}_{12}}{\partial w1} & \frac{\partial\tilde{\theta}_{12}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{13}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{13}}{\partial u1} & \frac{\partial\tilde{\theta}_{13}}{\partial u2} & \frac{\partial\tilde{\theta}_{13}}{\partial u3} & \frac{\partial\tilde{\theta}_{13}}{\partial w1} & \frac{\partial\tilde{\theta}_{13}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{21}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{21}}{\partial u1} & \frac{\partial\tilde{\theta}_{21}}{\partial u2} & \frac{\partial\tilde{\theta}_{21}}{\partial u3} & \frac{\partial\tilde{\theta}_{21}}{\partial w1} & \frac{\partial\tilde{\theta}_{21}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{22}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{22}}{\partial u1} & \frac{\partial\tilde{\theta}_{22}}{\partial u2} & \frac{\partial\tilde{\theta}_{22}}{\partial u3} & \frac{\partial\tilde{\theta}_{22}}{\partial w1} & \frac{\partial\tilde{\theta}_{22}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{23}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{23}}{\partial u1} & \frac{\partial\tilde{\theta}_{23}}{\partial u2} & \frac{\partial\tilde{\theta}_{23}}{\partial u3} & \frac{\partial\tilde{\theta}_{23}}{\partial w1} & \frac{\partial\tilde{\theta}_{23}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{31}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{31}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{31}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{31}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{31}}{\partial u1} & \frac{\partial\tilde{\theta}_{31}}{\partial u2} & \frac{\partial\tilde{\theta}_{31}}{\partial u3} & \frac{\partial\tilde{\theta}_{31}}{\partial w1} & \frac{\partial\tilde{\theta}_{31}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{32}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{32}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{32}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{32}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{32}}{\partial u1} & \frac{\partial\tilde{\theta}_{32}}{\partial u2} & \frac{\partial\tilde{\theta}_{32}}{\partial u3} & \frac{\partial\tilde{\theta}_{32}}{\partial w1} & \frac{\partial\tilde{\theta}_{32}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{33}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{33}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{33}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{33}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{33}}{\partial u1} & \frac{\partial\tilde{\theta}_{33}}{\partial u2} & \frac{\partial\tilde{\theta}_{33}}{\partial u3} & \frac{\partial\tilde{\theta}_{33}}{\partial w1} & \frac{\partial\tilde{\theta}_{33}}{\partial w2}
\end{vmatrix}$$

$$\begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Jacobian

$$\left|\begin{matrix}
\frac{\partial\tilde{\theta}_{11}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{11}}{\partial u1} & \frac{\partial\tilde{\theta}_{11}}{\partial u2} & \frac{\partial\tilde{\theta}_{11}}{\partial u3} & \frac{\partial\tilde{\theta}_{11}}{\partial w1} & \frac{\partial\tilde{\theta}_{11}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{12}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{12}}{\partial u1} & \frac{\partial\tilde{\theta}_{12}}{\partial u2} & \frac{\partial\tilde{\theta}_{12}}{\partial u3} & \frac{\partial\tilde{\theta}_{12}}{\partial w1} & \frac{\partial\tilde{\theta}_{12}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{13}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{13}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{13}}{\partial u1} & \frac{\partial\tilde{\theta}_{13}}{\partial u2} & \frac{\partial\tilde{\theta}_{13}}{\partial u3} & \frac{\partial\tilde{\theta}_{13}}{\partial w1} & \frac{\partial\tilde{\theta}_{13}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{21}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{21}}{\partial u1} & \frac{\partial\tilde{\theta}_{21}}{\partial u2} & \frac{\partial\tilde{\theta}_{21}}{\partial u3} & \frac{\partial\tilde{\theta}_{21}}{\partial w1} & \frac{\partial\tilde{\theta}_{21}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{22}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{22}}{\partial u1} & \frac{\partial\tilde{\theta}_{22}}{\partial u2} & \frac{\partial\tilde{\theta}_{22}}{\partial u3} & \frac{\partial\tilde{\theta}_{22}}{\partial w1} & \frac{\partial\tilde{\theta}_{22}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{23}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{23}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{23}}{\partial u1} & \frac{\partial\tilde{\theta}_{23}}{\partial u2} & \frac{\partial\tilde{\theta}_{23}}{\partial u3} & \frac{\partial\tilde{\theta}_{23}}{\partial w1} & \frac{\partial\tilde{\theta}_{23}}{\partial w2} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{matrix}\right|$$

$$\begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Jacobian

$$\begin{vmatrix}
\frac{\partial\tilde{\theta}_{11}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{11}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{11}}{\partial u1} & \frac{\partial\tilde{\theta}_{11}}{\partial u2} & \frac{\partial\tilde{\theta}_{11}}{\partial u3} & \frac{\partial\tilde{\theta}_{11}}{\partial w1} & \frac{\partial\tilde{\theta}_{11}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{12}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{12}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{12}}{\partial u1} & \frac{\partial\tilde{\theta}_{12}}{\partial u2} & \frac{\partial\tilde{\theta}_{12}}{\partial u3} & \frac{\partial\tilde{\theta}_{12}}{\partial w1} & \frac{\partial\tilde{\theta}_{12}}{\partial w2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\partial\tilde{\theta}_{21}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{21}}{\partial u1} & \frac{\partial\tilde{\theta}_{21}}{\partial u2} & \frac{\partial\tilde{\theta}_{21}}{\partial u3} & \frac{\partial\tilde{\theta}_{21}}{\partial w1} & \frac{\partial\tilde{\theta}_{21}}{\partial w2} \\
\frac{\partial\tilde{\theta}_{22}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{22}}{\partial u1} & \frac{\partial\tilde{\theta}_{22}}{\partial u2} & \frac{\partial\tilde{\theta}_{22}}{\partial u3} & \frac{\partial\tilde{\theta}_{22}}{\partial w1} & \frac{\partial\tilde{\theta}_{22}}{\partial w2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{vmatrix}$$

$$\begin{pmatrix} \tilde{\theta}_{11} & \tilde{\theta}_{12} & \tilde{\theta}_{13} \\ \tilde{\theta}_{21} & \tilde{\theta}_{22} & \tilde{\theta}_{23} \\ \tilde{\theta}_{31} & \tilde{\theta}_{32} & \tilde{\theta}_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Jacobian

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_{11} & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & -\theta_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{21}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{21}}{\partial u1} & \frac{\partial\tilde{\theta}_{21}}{\partial u2} & \frac{\partial\tilde{\theta}_{21}}{\partial u3} & \frac{\partial\tilde{\theta}_{21}}{\partial w1} & \frac{\partial\tilde{\theta}_{21}}{\partial w2} \\ \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{11}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{12}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{21}} & \frac{\partial\tilde{\theta}_{22}}{\partial\theta_{22}} & \frac{\partial\tilde{\theta}_{22}}{\partial u1} & \frac{\partial\tilde{\theta}_{22}}{\partial u2} & \frac{\partial\tilde{\theta}_{22}}{\partial u3} & \frac{\partial\tilde{\theta}_{22}}{\partial w1} & \frac{\partial\tilde{\theta}_{22}}{\partial w2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\begin{pmatrix} \widetilde{\theta}_{11} & \widetilde{\theta}_{12} & \widetilde{\theta}_{13} \\ \widetilde{\theta}_{21} & \widetilde{\theta}_{22} & \widetilde{\theta}_{23} \\ \widetilde{\theta}_{31} & \widetilde{\theta}_{32} & \widetilde{\theta}_{33} \end{pmatrix} =$$

$$\begin{pmatrix} (1-w_1)\theta_{11} & (1-w_1)\theta_{12} & w_1 \\ (1-w_2)\theta_{21} & (1-w_2)\theta_{22} & w_2 \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Jacobian

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_{11} & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & -\theta_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & -\theta_{21} \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & -\theta_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_{11} & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & -\theta_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & -\theta_{21} \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & -\theta_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-w_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-w_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1-w_1)^2(1-w_2)^2$$