A STATISTICAL APPROACH TO IMAGE WARPING

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Question 1: Can we superimpose microscope images?

brightfield

DIC

phase contrast
Question 2: Can we distinguish between species of fish?

haddock

whiting
Question 3: Are the *E. coli* samples already in database?

Pulsed-field gel electrophoresis (PFGE)
Who is this?
Image morphing

© http://en.wikipedia.org/wiki/Morphing
These are all examples of IMAGE WARPING

\[ Y_{x} \sim \mu f(x) \quad \text{OR} \quad Y_{(i,j)} \sim \mu f(i,j) \quad \text{where} \quad f(i,j) = (k, l) \]

‘\( \sim \)’ is a matching criterion

\( f \) is warping function, either parametric or with distortion penalty
Polynomial warps

Translation
\[ k = i + a_{00} \]
\[ l = j + b_{00} \]

Translation + Dilation
\[ k = ci + a_{00} \]
\[ l = cj + b_{00} \]

Translation + Rotation
\[ k = i \cos \theta + j \sin \theta + a_{00} \]
\[ l = -i \sin \theta + j \cos \theta + b_{00} \]

Procrustes
\[ k = ci \cos \theta + cj \sin \theta + a_{00} \]
\[ l = -ci \sin \theta + cj \cos \theta + b_{00} \]

Affine
\[ k = a_{10}i + a_{01}j + a_{00} \]
\[ l = b_{10}i + b_{01}j + b_{00} \]

Bilinear
\[ k = a_{10}i + a_{01}j + a_{11}ij + a_{00} \]
\[ l = b_{10}i + b_{01}j + b_{11}ij + b_{00} \]

Perspective
\[ k = (a_{10}i + a_{01}j + a_{00})/(c_{10}i + c_{01}j + 1) \]
\[ l = (b_{10}i + b_{01}j + b_{00})/(c_{10}i + c_{01}j + 1) \]

Polynomial
\[ k = \sum \sum a_{mn}i^{m}j^{n} \]
\[ l = \sum \sum b_{mn}i^{m}j^{n} \]
Parametric warping between species of fish:

*Argyropelecus olfersi.*  
*Sternoptyx diaphana.*  
*Scarus sp.*  
*Pomacanthus.*

D’Arcy Thompson, *On Growth and Form* (1917)
Distortion penalties have been motivated by:

- elastic membranes
- optical or fluid flow
- diffusion
- Markov random fields
- thin plate splines (≡ kriging)
An example of thin plate splines:

Venus → Venus warped → average

Arad et al. (*CVGIP: Graphical Models and Image Processing*, 1994)
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0. Introduction
1. Matching criteria
2. Distortion penalties
3. Algorithms
4. Summary
1. Matching criteria

Question 1: Can we superimpose microscope images?

The Fourier representation of $Y$ is

$$Y_x = \sum_{\omega} A_{\omega} \cos \left( \theta_{\omega} + 2\pi \omega^T x \right) \quad \forall x \in X.$$
Consider a second image, $\mu$, and difference the phases.
We propose a Fourier-von Mises model to estimate translation $\alpha$

$$\theta_{\omega} - \theta_{\omega}^{(\mu)} \sim \mathcal{M}(2\pi \omega^T \alpha, \kappa_{\omega})$$

where precision

$$\kappa_{\omega} \propto (A_{\omega})^{\gamma_1} (A_{\omega}^{(\mu)})^{\gamma_2} \exp \left[ \gamma_3 |\omega| + \gamma_4 |\omega|^2 \right]$$
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The log-likelihood

$$\mathcal{L} \propto \sum_\omega \kappa_\omega \cos (\theta_\omega - \theta_\omega^{(\mu)} + 2\pi \omega^T \alpha)$$

generalises the cross-covariance

$$C(\alpha) = \sum_x \mu_x Y_{x+\alpha} = \sum_\omega A_\omega A_\omega^{(\mu)} \cos (\theta_\omega - \theta_\omega^{(\mu)} + 2\pi \omega^T \alpha)$$

and the phase correlation

$$C'(\alpha) = \sum_\omega \cos (\theta_\omega - \theta_\omega^{(\mu)} + 2\pi \omega^T \alpha)$$

for special cases $\gamma = (1, 1, 0, 0)$ and $\gamma = (0, 0, 0, 0)$
Alignment of multiple \( \frac{1}{4} \times \frac{1}{4} \) subimages

<table>
<thead>
<tr>
<th>Similarity criterion</th>
<th>s.d.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha_1 )</td>
</tr>
<tr>
<td>Covariance</td>
<td>8.0</td>
</tr>
<tr>
<td>Phase-correlation</td>
<td>11.4</td>
</tr>
<tr>
<td>Fourier-von Mises log-likelihood</td>
<td>0.7</td>
</tr>
</tbody>
</table>
Pseudo-coloured composite image
Landsat Thematic Mapper (TM) images of east coast of Scotland
Phase differences between Fourier transforms of pairs of bands
Phase difference between bands 1 and 2

The linear trend in phase differences is affected by aliasing at high frequencies
We have developed a method for aligning images, which takes account of aliasing.

We use a parametric model for the power- and cross-spectra of the multivariate, stochastic process in continuous-space.

A side benefit is improved interpolation: de-aliasing using information from other bands.
Tay bridge

fields

original cubic interpolation new result
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Question 2: Can we distinguish between species of fish?

one ‘species’

another ‘species’
We estimate the warp, \( f = (f_1, f_2) \), to minimise a penalised sum of squares

\[
P = \sum_x \left[ Y_x - \mu f(x) \right]^2 + \lambda D(f, \mathcal{C})
\]

where \( D \) is a distortion penalty such that

\[
D(f, \mathcal{C}) = 0 \quad \forall f \in \mathcal{C} \\
> 0 \quad \forall f \notin \mathcal{C}.
\]

For the commonly used thin-plate-spline penalty

\[
D = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \int_{\mathbb{R}^2} \left( \frac{\partial^2 f_i}{\partial x_j \partial x_k} \right)^2 dx
\]

\( \mathcal{C} \) is the set of affine transformations
For shape comparisons, we wish to restrict $C$ to Procrustes similarity transformations ($S$).

We do this by modifying an existing (base) distortion penalty ($D_B$):

$$D(f, S) = \min_{g \in S} D_B(f - g)$$

If

$$D_B(f) = \sum_{i=1}^{2} \sum_{j=1}^{2} \iint \left( \frac{\partial f_i}{\partial x_j} \right)^2 dx$$

then

$$D(f, S) = D_B(f) - 2n_1n_2(\tilde{\alpha}_{11}^2 + \tilde{\alpha}_{12}^2),$$

where the images are of size $n_1 \times n_2$ and

$$\tilde{\alpha}_{11} = \frac{1}{2n_1n_2} \iint \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx,$$

$$\tilde{\alpha}_{12} = \frac{1}{2n_1n_2} \iint \left( \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \right) dx.$$
Using $D(f, S)$, all values of $P$ for within "species" comparisons are less than those for between "species" comparisons:

Whereas, if the thin-plate-spline penalty is used, within "species" values of $P$ are no smaller than between "species" comparisons:

$(\lambda$ chosen to maximise studentised difference between sets of comparisons)
We formed average fish by warping into alignment 8 haddock and 8 whiting

average haddock

average whiting
Distances between average fish and 16 used in training plus 4 more (circled)
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Question 3: Are the \emph{E. coli} samples already in database?

We seek the warping function \((f)\) and lane labels \((l)\) that minimise

\[
C(f, l) = \sum_i \sum_j \left[ (Y_{ij} - \mu_{i+f_{ij},l_{ij}})^2 + \lambda_1 (f_{ij} - f_{i-1,j})^2 + \lambda_2 (f_{ij} - f_{i,j-1})^2 \right]
\]
We can use the three generalisations of DP:

**Gel 1**

**CPU (sec)**

<table>
<thead>
<tr>
<th>IDP</th>
<th>stoch-IDP</th>
<th>FB-SA</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>100</td>
<td>1000</td>
</tr>
<tr>
<td>100</td>
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<td>10000</td>
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**Gel 2**

**CPU (sec)**

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$Y_{\bar{f}_{ij}^{-1},j}$, where $f^{-1}$ is an inverse function in the first index, defined such that $f_{\bar{f}_{ij}^{-1},j} \equiv i$.
Re-ordered

class 1        class 2        class 3        class 4        new class

Arrowed columns visually identified as belonging to *E. coli* strains not in database
4. Summary

We have considered some aspects of the broad subject of image warping:

- Fourier matching criteria
- null-set distortion penalties
- generalisations of dynamic programming

With applications in:

- microscopy
- digital photography
- bioinformatics

For further details, see papers on
http://www.bioss.ac.uk/staff/chris.html