Some conditional extremes of a Markov Chain

Postgraduate forum on 16th January 2003

Adam Butler

In collaboration with...

- Janet Heffernan
- Jonathan Tawn
- Roger Flather (Proudman Oceanographic Laboratory)
- David Blackwell (Proudman Oceanographic Laboratory)

Conditional univariate extremes

(1.1) Probabilistic aspects

...

- Consider a univariate random variable X_1.
- Consider the distribution of $\text{Lim}_{x_1 \to \infty}(X_1 - x_1 | X_1 > x_1)$.
- Under certain regularity conditions, it may be shown that, *whatever the distribution of* X_1, this random variable has a generalized pareto distribution (GPD) with scale parameter $\sigma(x_1)$ and shape parameter ξ.
- GDP(σ, ξ) has distribution function

$$H(x_1) = 1 - \left(1 + \frac{\xi x_1}{\sigma}\right)^{-1/\xi}$$

- This is all well-known - for mathematical details, see Leadbetter et al. (1983); for an overview, see Coles (1999).

...

(1.2) Statistical aspects

. . .

- We are interested in modelling extreme values of a process.
- Assume that the asymptotic results holds approximately sub-asymptotically.
- Model exceedances above a high marginal threshold as being realisations from a GPD distribution.
- Threshold choice may be motivated by physical considerations.
- Diagnostic methods exist to aid threshold selection.
- Invalid if threshold exceedances are strongly dependent; should use peaks over threshold (POT) method.
- This is not the only approach to modelling extremes, but is probably the most efficient.

. . .

Conditional multivariate extremes

(2.1) Basic probabilistic ideas

...

- We will follow (and later develop) the approach of Heffernan and Tawn (2003).
- Assume $X = (X_1, \ldots, X_d)$ is an arbitrary d dimensional multivariate distribution.
- Throughout, $d \geq 2$ is taken finite.
- Let $F(x) = \mathbf{P}(X_1 = x_1, \ldots, X_d = x_d)$ be the distribution function of X.
- Consider limit distributions of the type $\text{Lim}_{x_1 \to \infty}(X_2, \ldots, X_d | X_1 = x_1)$.
- This is a $(d-1)$ dimensional distribution.
- For convenience, assume throughout that all of the marginal distributions of $X_1, \ldots, X_d$ are standard Gumbel, so that

  $$F(x_i) = \exp(-\exp(-x_i))$$

  for $i = 1, \ldots, d$.
- We therefore focus only upon the dependence aspects of X.

...

(2.2) Renormalization

...

- There is no reason to suppose that the limit distribution will also have standard Gumbel marginal distributions.
- Extending this, there is no reason to suppose that the margins of the limit distribution will be nondegenerate.
- In fact, as we shall see, the margins of the limit distribution will frequently be degenerate.
- We are only really interested interested in limit distributions which have non-degenerate marginal distributions.

...

- This can only be achieved by renormalization of X.
- We consider the renormalized random variables $Z_{|1} = (Z_{1|1}, \ldots, Z_{d|1})$, where $Z_{1|1} = X_1$ and

$$Z_{j|i} = \frac{Y_j - a_{j|1} y_i}{y_i^{b_{j|1}}}$$

  for $j \geq 2$.
- We are now interested in distributions of the form $\mathrm{Lim}_{z_{1|1} \to \infty}(Z_{2|1}, \ldots, Z_{d|1} | Z_1 = z_{1|1})$.
- Let $G_{|1}$ denote the corresponding distribution function, such that ... .
- The constants $a_{j|1} \in [0, 1]$ and $b_{j|1} \in [0, 1)$ are *chosen* in such a way as to ensure that the limit distribution $G_{|1}$ has all non-generate margins.

...

(2.3) General mathematical properties

...

- There are $2(d-1)$ renormalization constants.
- Renormalization constants of the form $a_{j|1} \in [0,1]$ and $b_{j|1} \in [0,1)$ need not exist.
- For example, renormalization constants of this form do not exist under perfect dependence or negative extremal dependence.
- However, for a very wide range of interesting examples, suitable renormalization constants *do* exist.
- If such constants do exist, they are unique (proof: Heffernan and Tawn, 2003).

...

- In general, nothing is known the form of the limit distribution $G_{|1}$.
- For different practically interesting examples, very different forms for $G_{|1}$ may be obtained.
- Let $G_{j|1}$ be the i^{th} marginal distribution of $G_{|1}$.
- $G_{|1}$ is asymptotically conditionally independent if $G_{|1}(z_{2|1}, \ldots, z_{d|1}) = \prod_{j=2}^{d} G_{j|1}(z_{j|1})$

...

(2.4) Multiple conditioning

$\dots$

- Now consider limit distributions of the kind

$$\text{Lim}_{z_k \to \infty}(Z_1, \dots, Z_{k-1}, Z_{k+1}, \dots, Z_d | Z_k = z_k),$$

  where $1 \leq k \leq d$.

- Extend all notation in the obvious way . . .

- What is the relationship between normalization constants (and limit distributions) which refer to *different* conditioning variables ?

- Weak extremal exchangability of X: if $a_{j|i} = a_{i|j}$ and $b_{j|i} = b_{i|j}$ for all $i, j \in (1, \dots, d)$ such that $i \neq j$.

- Strong extremal exchangability of X: if $G_{j|i}(z) = G_{i|j}(z)$ for all $i, j \in (1, \dots, d)$ such that $i \neq j$, and for all $z \in \mathbf{R}$.

$\dots$

(2.5) Major examples

...

We assume that the joint distribution of X, $F(x) = \mathbf{P}(X_1 \leq x_1, \ldots, X_d \leq x_d)$, is known; the following results are from Heffernan and Tawn (2003).

Near independence

- Near independence: $a_{j|i} = b_{j|i} = 0$ for $\{i \in (1, \ldots, n), j \in (1, \ldots, n) : i \neq j\}$.
- Gumbel marginal distributions in $G_{|1}$.
- Examples: Independence, multivariate morgenstern.

Asymptotic dependence

- Asymptotic dependence: $a_{j|i} = 1, b_{j|i} = 0$ for $\{i \in (1, \ldots, n), j \in (1, \ldots, n) : i \neq j\}$.
- Examples: Multivariate extreme value distribution.

Intermediate extremal dependence

- Multivariate normal distribution, with $0 < \rho_{ij} < 1$ for all $\{i, j \in (1, \ldots, n) : i \neq j\}$: $b_{j|i} = 0.5$, $0 < a_{j|i} < 1$.
- Inverted multivariate extreme value distribution: $a_{j|i} = 0, 0 < b_{j|i} < 1$.

...

(2.6) First-order Markov structure

$\ldots$

Assume that the distribution of (X_1, X_2) is known, that the distribution of (X_2, X_3) is known and that we have First-order Markov structure, so that

$$\mathbf{P}(X_2 = x_2, X_3 = x_3 | X_1 = x_1) = \mathbf{P}(X_3 = x_3 | X_2 = x_2)\mathbf{P}(X_2 = x_2 | X_1 = x_1).$$

What are $a_{3|1}$ and $b_{3|1}$?

(X_1, X_2)	$a_{2\|1}$	$b_{2\|1}$	(X_2, X_3)	$a_{3\|2}$	$b_{3\|2}$	$a_{3\|1}$	$b_{3\|1}$
Independence	0	0	Independence	0	0	0	0
Morgenstern	0	0	Morgenstern	0	0	0	0
BVN(ρ_{12})	ρ_{12}	0.5	BVN(ρ_{23})	ρ_{23}	0.5	$\rho_{12}\rho_{23}$	0.5
Inverted BEVL(ψ_{12})	0	ψ_{12}	Inverted BEVL(ψ_{23})	0	ψ_{23}	0	$\psi_{12}\psi_{23}$
BEVL(ψ_{12})	1	0	BEVL(ψ_{23})	1	0	1	0
Independence	0	0	BEVL(ψ_{23})	0	ψ_{23}	0	0
Morgenstern	0	0	Perfect dependence	NA	NA	0	0
Inverted BEVL(ψ_{12})	0	ψ_{12}	BVN(ρ_{23})	ρ_{23}	0.5	?	?
BVN(ρ_{12})	ρ_{12}	0.5	Inverted BEVL(ψ_{23})	0	ψ_{23}	?	?

$\ldots$

(2.7) Statistical modelling

...

Single site

- Begin by considering conditioning upon each site in turn.
- Assume that asymptotic limit theory may be applied for extreme but finite values of x_1 - for values above some dependence threshold.
- Incorporate marginal information (semiparametrically).
- Impose sensible restraints upon the relationships between the normalizing constants (e.g. perhaps $a_{3|1} = \phi a_{2|1}$, for some constant $\phi > 0$).
- Impose "arbitrary" restrictions upon $G_{|1}$: Gaussianity, and independence (Heffernan and Tawn, 2003).

All sites

- Model the entire extremal dependence structure of X, by constructing a model in which we condition upon each component of X in turn being large.
- Impose further constraints upon the normalizing constants, e.g. weak extremal exchangability.

...

Environmental application

(3.1) Environmental background

. . .

Climate change and extreme sea levels

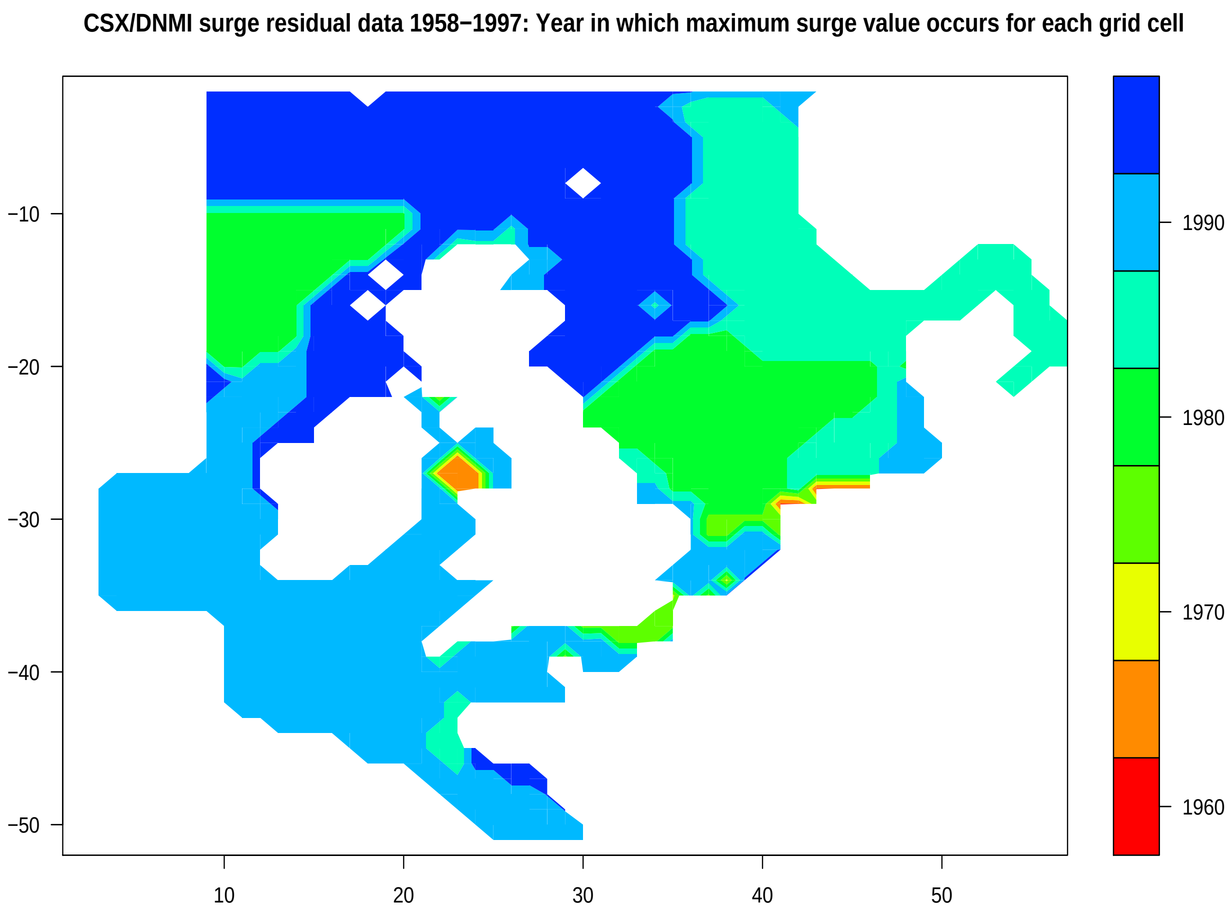
- Sea-levels: mean sea level, waves, tides, surge, tide-surge interaction, surge residuals
- Climate change: Impact upon mean sea levels
- Impact upon extremal properties of surge residuals
- Attribution to anthropogenic sources

Data

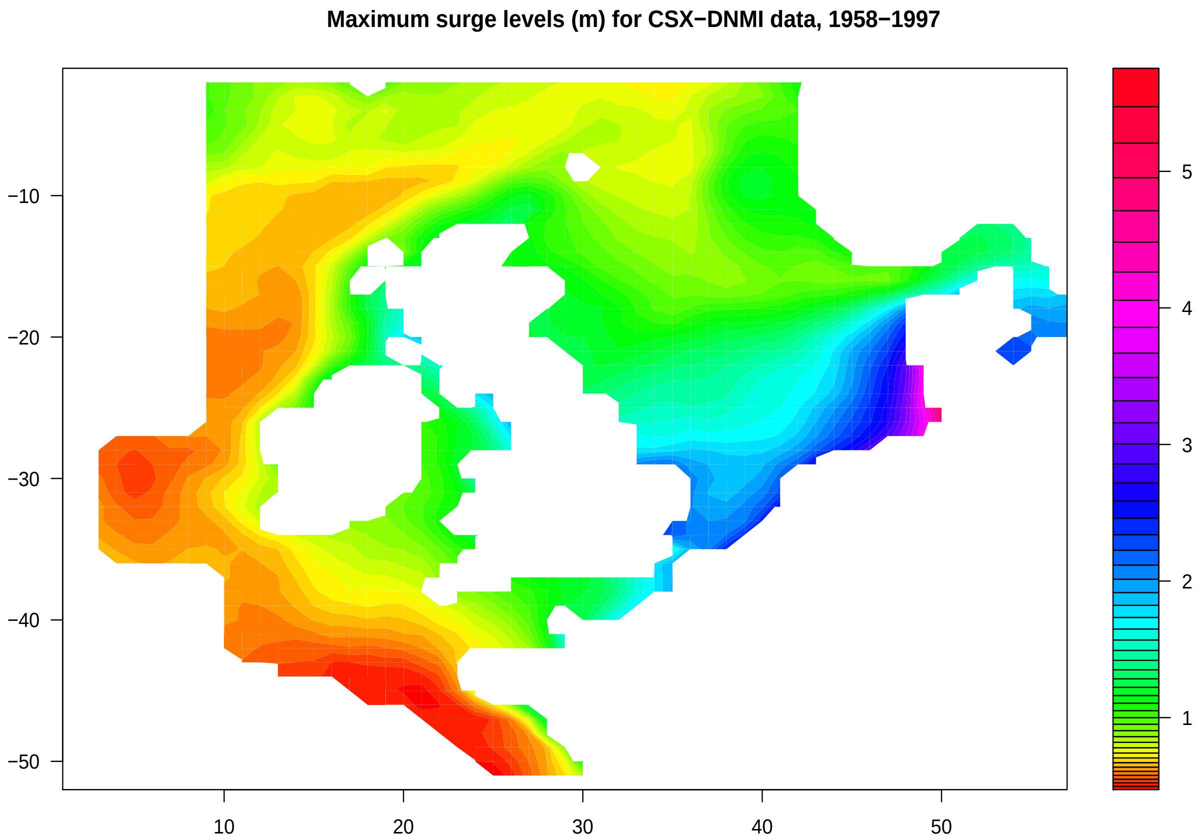
- Output from a hydrodynamical model (CSX), run using meteorological inputs.
- Hourly output, for each of 1300 grid cells on the European Continental Shelf.
- Meterological inputs generated using a weather model which assimilates observational data (DNMI) or using a climate model (ECHAM4).
- Extremal trend in CSX/DNMI data (1955-1997): attributable to increasing CO_2 or creeping inhomogeneities ?
- Differences between extremal properties of CSX/ECHAM4 data generated under different CO_2 scenarios ? Causal ! Maybe...

. . .

(3.2) Preliminary evidence for trend in CSX/DNMI data



(3.3) Preliminary evidence for spatial structure



(3.4) Spatial extreme value model

. . .

We focus *specifically* upon the extremes. We construct a multivariate model for extreme values, with each site corresponding to one dimension. Our model incorporates -

- Marginal structure - conditional univariate extremes;
- Temporal evolution - independent observations, possible linear trend;
- Spatial coherence - parameter surface modelling via local likelihood;
- Residual spatial dependence - conditional multivariate extremes;
- Spatial unity - dependence and coherence are *related*.

. . .

Current methods only incorporate marginal structure.

Possible benefits from a spatial analysis -

- Improved precision in marginal estimation and extrapolation at a single site (flood defense, offshore engineering);
- Possibility for region-level estimation and testing (insurance);
- Information about spatial structure (oceanography, climate change detection);
- Hydrodynamical model experiment design (oceanography).

. . .

Images

- Damage caused by the 1953 coastal floods in Zealand (Netherland Photo Archive, http://www.nfa.nl/index.ht
- Pictures of North Sea oil rigs, by Derek Mackay and Ewan Mitchell, http://www.btinternet.com/ derek.macka

. . .